Entanglement Optimization for Pairs of Qubits

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Local Operations enhancing the entanglement of bipartite quantum states are of great interest in quantum information processing. Subject of this paper are local selective operations acting on single copies of states. Such operations can lead to larger entanglement with respect to a certain measure as studies before by the Horodeckis and A. Kent et al. [1–3]. We present a complete characterisation of all local operations yielding optimal entanglement for pairs of qubits, extending former results of A. Kent et al. We introduce a new technique for the classification of states according to their behaviour under entanglement optimizing operations, using the entanglement properties of the support of density matrices.

The key resource of quantum information theory is quantum entanglement. It is of vital importance to find a measure for entanglement and investigate how it can be increased for a given state. Many applications of quantum information theory, like teleportation, dense coding or quantum cryptography, require maximally entangled states of two qubits shared by two distinct parties, traditionally called Alice and Bob. Maximally entangled Pairs, however, have to be prepared in a common quantum process at a certain place. In order to share them between Alice and Bob at least one of the two particles must be sent through a quantum channel (e.g. an optical fibre). During this process interaction with the environment leads to a loss of entanglement, the state evolves to a non-maximally entangled mixed state.

Therefore purification, also called distillation, becomes necessary: a process which increases the entanglement of given pairs by local operations and classical communication (LQCC) performed by Alice and Bob. The first purification protocol was presented by Bennett et al. [5]. It involves local operations acting on many pairs, called collective operations. Using such a collective scheme one can obtain pure maximally entangled particles from any given two-spin-$\frac{1}{2}$-state, even from mixed ones.

Recent publications raised the question, if it is also possible to purify arbitrary two-spin-$\frac{1}{2}$-states if only non-collective operations are allowed, i.e. operations that act on each pair individually. It was shown that this is possible only for pure states. Mixed states of two qubits cannot be purified to maximally entangled states by local operations. Nevertheless there are mixed states whose entanglement can be increased by such protocols. For some of these states it is even possible to reach maximal entanglement, but only with vanishing probability, i.e. there is no limit for the entanglement that can be distilled from these states. This process was introduced by Horodecki et al. [1] and called *quasi-distillability*.

In this letter we classify states of two qubits with respect to their maximal distillable entanglement. First we present the entanglement properties of the support of the considered density matrices, which will lead to the classification given below. The proofs will be given in the last part of this letter.

We use the following facts (see [6,2,3]):

- The considered distillation protocols involve LQCC's of a special form: after the operation the state ρ takes the form
$$\Theta(\rho) = \frac{A \otimes B \rho A \otimes B}{\mathrm{Tr}(A \otimes B \rho A \otimes B)} \tag{1}$$
- The entanglement of formation (E_F) of a state of two qubits can be calculated by the formula of Wootters as
$$E_F(\rho) = H\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right)$$
with $H(p) = p\log_2 p - (1-p)\log_2(1-p)$
and the concurrence C, which is given by $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, the difference of the eigenvalues λ_i of $\tilde{\rho}\rho = \sigma_2 \otimes \sigma_2 \bar{\rho} \sigma_2 \otimes \sigma_2 \rho$, where the eigenvalues are taken in decreasing order.
- Linden, Massar and Popescu have shown in [7] that under operations of the form (1) the eigenvalues λ_i of the matrix $\tilde{\rho}\rho$ transform as
$$\lambda_i^{'} = c_1 \lambda_i \tag{2}$$
with a factor c_1 that is independent of i.

  As a consequence the ratio of the eigenvalues is constant under local operations.

We will show that the main qualitative feature of density matrices with respect to entanglement optimization procedures is the number of product vectors in the *support* of this matrix.

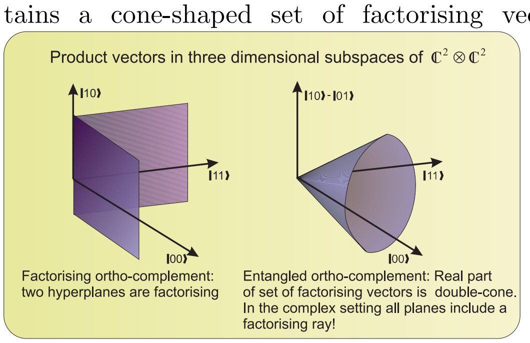
The *support* of a density matrix is the orthogonal complement of its *kernel*, i.e. for selfadjoint matrices identical with its *range*. We can also think of the support as the linear span of the eigenvectors with non-vanishing eigenvalues. This subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ determines a subset of the state space containing all states having support

included in this subspace. Such a subset of state space is called a *face* in convex analysis. The different faces of a state space over a tensor product Hilbert space can be characterised firstly by the dimension of the respective subspace (i.e. the rank of the generic elements) and secondly by the structure of the subset of product vectors in this subspace. The first property is invariant under all linear isomorphisms of the Hilbert space. The second one is invariant under *factorising* linear isomorphisms.

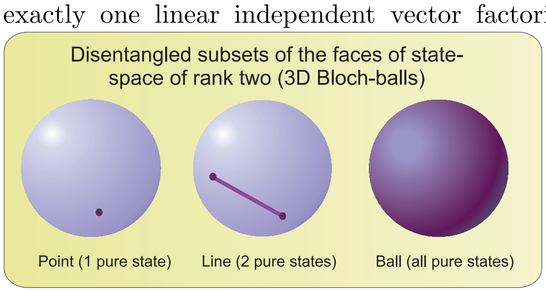
In the simplest case of two qubits a complete classification can be done. For that purpose we have a look at all possible subspaces $U \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$ (see also [4] for some details):

A. The case $\dim U = 4$ is trivial. There is only one such subspace: $\mathbb{C}^2 \otimes \mathbb{C}^2$ itself.

B. For $\dim U = 3$ there are two cases, easily characterised by their one dimensional orthogonal complement (the *kernel* of the respective matrices). Either the complement is factorising and the subspace contains two factorising hyperplanes, or the complement is entangled and the subspace contains a cone-shaped set of factorising vectors.



C. $\dim U = 2$ offers three possibilities: a) the whole subspace is factorising, b) exactly two linear independent vectors are factorising and c) exactly one linear independent vector factorises.



D. $\dim U = 1$ is trivial again. Either it factorises, or not.

These properties of the support of density matrices will now be shown to be the only features determining the behaviour under local transformations with respect to entanglement optimization.
We state our results as follows:

Let ρ be an entangled state on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Set

$$M_\rho := \sup\{E_F(\Theta(\rho)) : \Theta \text{ local operation } \}.$$

If ρ is Bell-diagonal, it has been shown in [2] that no local operation can increase the Entanglement of Formation $E_F(\rho)$ of the state, i.e. $M_\rho = E_F(\rho)$. Thus we assume here that ρ be **not** Bell-diagonal. Then one of the following holds true:

I. $\operatorname{rank} \rho = 4$. As seen in [2] there exists a local operation Ξ, such that $E_F(\Xi(\rho)) = M_\rho < 1$ and $\Xi(\rho)$ is Bell-diagonal. We call this property of the state *incomplete distillability.*

II. $\operatorname{rank} \rho = 3$. $M_\rho < 1$. Either (i.) the kernel of ρ contains no product vector and a local operation Ξ exists, such that $E_F(\Xi(\rho)) = M_\rho$ and $\Xi(\rho)$ is Bell-diagonal, i.e. ρ is *incompletely distillable.* Or (ii.) the kernel of ρ is the linear span of a product vector and there is no local operation with $E_F(\Xi(\rho)) = M_\rho$. In this case we can find a sequence of local operations $\{\Xi_n\}_n$, with $\sigma = \lim_{n\to\infty} \Xi_n(\rho)$ Bell-diagonal and $E_F(\sigma) = M_\rho$. These states will be called *incompletely quasi-distillable.*

III. $\operatorname{rank} \rho = 2$. Either (i.) the support of ρ contains exactly one linear independent product vector. In this case $M_\rho = 1$, but there is no local operation with $E_F(\Xi(\rho)) = 1$. Instead we find a sequence of local operations $\{\Xi_n\}_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} \Xi_n(\rho)$ a Bell-state (*i.e.* ρ is *quasi-distillable*). Or (ii.) the support of ρ contains exactly two linear independent product vectors and it exists a local operation Ξ, such that $E_F(\Xi(\rho)) = M_\rho$ and $\Xi(\rho)$ is Bell-diagonal, i.e. the state is *incompletely distillable.*

IV. $\operatorname{rank} \rho = 1$, i.e. the state is pure: $M_\rho = 1$ Then ρ ist *distillable*, i.e. there is a local operation Ξ, such that $\Xi(\rho)$ is a Bell-state.

In order to prove our programme we have a look at the previously existing results. Kent et al. have shown in [3] that states can be brought to a Bell-diagonal form with optimal entanglement, as long as the quantity $\operatorname{Tr}(A \otimes B\rho A^\dagger \otimes B^\dagger)$ cannot be zero for those states. Nevertheless for states that do not fulfil this assumption, like quasi-distillable states, an essential continuity argument fails. For that reason we will now characterize quasi-distillable and incompletely quasi-distillable states, which will lead to the characterization given above.
Lemma: A quasi-distillable or incompletely quasi-distillable state has a factorising vector in its kernel.

Proof: If ρ is quasi-distillable or incompletely quasi-distillable, then the probability for reaching the product $\Xi_n(\rho)$ tends to 0 as $n \to \infty$.

In the following we will use the operators $\tilde{A}_n \otimes \tilde{B}_n := \frac{A_n \otimes B_n}{\|A_n \otimes B_n\|}$, which produce the same state $\Xi_n(\rho)$. The probability of getting the state $\Xi_n(\rho)$ is given by

$$p_n := \mathrm{Tr}(\tilde{A}_n \otimes \tilde{B}_n \rho \tilde{A}_n^\dagger \otimes \tilde{B}_n^\dagger).$$

Since ρ is quasi-distillable, it follows, that

$$\lim_{n\to\infty} \mathrm{Tr}(\tilde{A}_n \otimes \tilde{B}_n \rho \tilde{A}_n^\dagger \otimes \tilde{B}_n^\dagger) = \lim_{n\to\infty} \mathrm{Tr}(\rho \tilde{A}_n^\dagger \tilde{A}_n \otimes \tilde{B}_n^\dagger \tilde{B}_n) = 0.$$

Since $\|\tilde{A}_n \otimes \tilde{B}_n\| = 1$ the set of these operators is compact and there exists a subsequence $\tilde{A}_{n_i} \otimes \tilde{B}_{n_i}$ that tends to a limit operation $\tilde{A} \otimes \tilde{B} := \lim_{i\to\infty} \tilde{A}_{n_i} \otimes \tilde{B}_{n_i}$ such that $\|\tilde{A} \otimes \tilde{B}\| = 1$. Then it follows that

$$\begin{aligned} &\lim_{i\to\infty} \mathrm{Tr}(\tilde{A}_{n_i} \otimes \tilde{B}_{n_i} \rho \tilde{A}_{n_i}^\dagger \otimes \tilde{B}_{n_i}^\dagger) \\ &= \mathrm{Tr}(\tilde{A} \otimes \tilde{B} \rho \tilde{A}^\dagger \otimes \tilde{B}^\dagger) \\ &= \mathrm{Tr}(\rho \tilde{A}^\dagger \tilde{A} \otimes \tilde{B}^\dagger \tilde{B}) = 0 \end{aligned}$$

The operators $\tilde{A}^\dagger \tilde{A}, \tilde{B}^\dagger \tilde{B}$ are positive and can in order of the spectral theorem be decomposed by

$$\tilde{A}^\dagger \tilde{A} = \sum_i a_i P_i, \qquad \tilde{B}^\dagger \tilde{B} = \sum_j b_j Q_j$$

with $a_i, b_j \in \mathbb{R}^+$ and projectors $P_i = |\phi_i\rangle\langle\phi_i|, Q_j = |\psi_j\rangle\langle\psi_j|$. The above equation then takes the form

$$\begin{aligned} &\mathrm{Tr}(\rho \tilde{A}^\dagger \tilde{A} \otimes \tilde{B}^\dagger \tilde{B}) = 0 \\ \Leftrightarrow &\sum_{i,j} \mathrm{Tr}(\rho a_i b_j P_i \otimes Q_j) = 0 \\ \Leftrightarrow &\sum_{i,j} a_i b_j \langle \phi_i \otimes \psi_j | \rho | \phi_i \otimes \psi_j \rangle = 0 \\ \Rightarrow &\langle \phi_i \otimes \psi_j | \rho | \phi_i \otimes \psi_j \rangle = 0 \quad \text{for all} \quad a_i, b_j, \quad a_i b_j \neq 0 \\ \Rightarrow &\rho^{\frac{1}{2}} |\phi_i \otimes \psi_j\rangle = 0 \quad \text{for all} \quad a_i, b_j, \quad a_i b_j \neq 0. \end{aligned}$$

The $|\phi_i \otimes \psi_j\rangle$ are then part of the kernel of $\rho^{\frac{1}{2}}$ for all a_i, b_j with $a_i b_j \neq 0$ and also part of the kernel of ρ. Since the operation $\tilde{A} \otimes \tilde{B}$ cannot be zero there has to be a product $a_i b_j \neq 0$ and therefore the corresponding product vector $|\phi_i \otimes \psi_j\rangle$ lies in the kernel of ρ. □

Proposition: A density matrix ρ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is quasi-distillable if and only if it has rank two and the support of ρ is spanned by a factorising vector $\phi_1 \otimes \phi_2$ and an orthogonal entangled vector ψ.

Proof: For any quasi-distillable state ρ there exists a sequence of operations $\{\Xi_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \Xi_n(\rho) = \rho_\infty$ where ρ_∞ is a Bell-state. Since the vector $\vec{\lambda}_\infty$ containing the eigenvalues of $\tilde{\rho}_\infty \rho_\infty$ is given by $(1,0,0,0)$ and by (2) the ratio of eigenvalues cannot change under the local operation Ξ_n, the vector $\vec{\lambda}_0$ belonging to the original state ρ must be of the form $(\lambda_0, 0, 0, 0)$, with $\lambda_0 \neq 0$, i.e. the rank of $\tilde{\rho}\rho$ is 1.

Since ρ is quasi-distillable, there is a factorising vector $\phi_3 \otimes \phi_4$ in its kernel. In the tensor basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ we can assume without loss of generality that $\phi_3 \otimes \phi_4$ have the form $|11\rangle$. The matrix form of ρ is in that case

$$\rho = \begin{pmatrix} 1-a-d & e & c & 0 \\ \bar{e} & a & b & 0 \\ \bar{c} & \bar{b} & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Since ρ ist positive, we have $a, d \geq 0$ and the eigenvalues of the matrix $\tilde{\rho}\rho$ are $(0, 0, (\sqrt{ad} - |b|)^2, (\sqrt{ad} + |b|)^2)$. Since $\tilde{\rho}\rho$ is of rank 1, we get $\sqrt{ad} = |b|$. This leads to

$$\rho = \begin{pmatrix} 1-a-d & e^{i\delta} c \sqrt{\frac{a}{d}} & c & 0 \\ e^{-i\delta} \bar{c} \sqrt{\frac{a}{d}} & a & e^{i\delta}\sqrt{ad} & 0 \\ \bar{c} & e^{-i\delta}\sqrt{ad} & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the condition $(1+a+d)d \leq |c|^2 \leq (1+a+d)d + \frac{d}{4(a+d)}$ has to be made for ρ to be positive.

The support of ρ is therefore the subspace spanned by the vectors $|00\rangle$ and $\frac{1}{\sqrt{a+d}}(\sqrt{a}|01\rangle + \sqrt{d}|10\rangle)$.

These states are indeed quasi-distillable, which can be shown by using the following protocol:

I. First Alice applies the local filtration $A = \mathrm{diag}(\sqrt{d}, \sqrt{a})$.

II. Now Alice and Bob apply the local operation given by

$$A_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{pmatrix}, B_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

This operation produces the state

$$\begin{aligned} \rho_n &= \frac{A_n \otimes B_n \rho' A_n \otimes B_n}{\mathrm{Tr}(A_n \otimes B_n \rho' A_n \otimes B_n)} \\ &= c_1 \begin{pmatrix} \frac{1-a-d}{1-a-d+2an^2)} & \frac{e^{i\delta} c}{(\frac{1-a-d}{n}+2an)} & \frac{c\sqrt{ad}}{(\frac{1-a-d}{n}+2an)d} & 0 \\ \frac{e^{-i\delta}\bar{c}}{(\frac{1-a-d}{n}+2an)} & \frac{a}{\frac{1-a-d}{n^2}+2a} & \frac{e^{i\delta} a}{\frac{1-a-d}{n^2}+2a} & 0 \\ \frac{\bar{c}\sqrt{ad}}{(\frac{1-a-d}{n}+2an)d} & \frac{e^{-i\delta} a}{\frac{1-a-d}{n^2}+2a} & \frac{a}{\frac{1-a-d}{n^2}+2a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

with $c_1 = \frac{1}{d(1+a-d)}$.

It can be shown that

$$\lim_{n\to\infty} \rho_n = \rho_\infty = |01\rangle + e^{i\lambda}|10\rangle.$$

This is a maximally entangled state and thus ρ is quasi-distillable. □

Proposition: A density matrix ρ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is incompletely quasi-distillable if and only if the kernel of ρ is spanned by a factorising vector $\phi_1 \otimes \phi_2$. The best

distillation product of ρ is in that case a Bell diagonal state of rank two.

Proof: Again we assume without loss of generality, that the state can be written as (3). We know from [2] that the entanglement of formation of a state can be further increased if and only if it is not Bell diagonal. An optimal distillation product must be thus Bell diagonal. From this and the fact that the distillation product must still have a factorising vector in its kernel, it follows that such a *final* state must be of the form

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & f & 0 \\ 0 & \bar{f} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which yields for $\sigma\tilde{\sigma}$ eigenvalues $(0, 0, (1/2+|f|)^2, (1/2-|f|)^2)$. Since by [3]the ratio of these eigenvalues cannot change under local operations, we see that

$$\frac{\sqrt{ad}+|b|}{\sqrt{ad}-|b|} = \frac{1/2+|f|}{1/2-|f|}$$

must be fulfilled. From this we get

$$\frac{|b|}{2\sqrt{ad}} = |f|.$$

The rank of σ is two whereas ρ has rank three, because it is neither distillable nor quasi-distillable. For that reason no local operation can transform ρ into the entangled state σ, since such a local operation would have to decrease the rank (thus being not one to one) and have entangled vectors in its range (see [2]). Nevertheless there is a sequence of operations which with probability tending to zero yields σ in the limit. For this purpose we use the operation elements

$$A_n := \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix} \quad B_n := \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

getting

$$\rho_n := A_n \otimes B_n \rho A_n \otimes B_n \tag{5}$$

$$= \begin{pmatrix} \frac{1-a-d}{n^4} & \frac{e}{n^3\sqrt{a}} & \frac{c}{n^3\sqrt{d}} & 0 \\ \frac{\bar{e}}{n^3\sqrt{a}} & \frac{1}{n^2} & \frac{b}{n^2\sqrt{ad}} & 0 \\ \frac{\bar{c}}{n^3\sqrt{d}} & \frac{\bar{b}}{n^2\sqrt{ad}} & \frac{1}{n^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

We easily see that the limit of the normalized operation (only leaving those terms proportional to $1/n^2$) is the state σ. □

Proposition: An entangled density matrix ρ on $\mathbb{C}^2\otimes\mathbb{C}^2$ of rank two is incompletely distillable if and only if the support of ρ is spanned by two factorising vectors $\phi_1\otimes\phi_2$ and $\psi_1\otimes\psi_2$.

Proof: Since ρ is entangled, there must be an entangled vector in its support. If there would be only one such vector, ρ would be quasi-distillable (see above). For that reason there must be *exactly* two linear independent factorising vectors in its support, i.e. no linear combination of these two factorises . Choosing without loss of generality $\phi_1 \otimes \phi_2 = |0\rangle|1\rangle$ we see that $\psi_1 \otimes \psi_2 = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$, where $\beta \neq 0$ and $\gamma \neq 0$. The invertible local operation

$$\Psi \mapsto \begin{pmatrix} 1 & \frac{-\alpha}{\beta} \\ 0 & \frac{1}{\beta} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\gamma} & 0 \\ \frac{-\delta}{\gamma} & 1 \end{pmatrix} \Psi$$

leaves $|01\rangle$ invariant, but transforms $\psi_1 \otimes \psi_2$ into $|10\rangle$. This means we can assume $\psi_1\otimes\psi_2 = |10\rangle$ without changing the properties of the state with respect to local transformations *qualitatively.* In this case, following the notation of the preceding proof, the density matrix reads:

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & \bar{b} & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The final argument follows the preceding proof, yielding a distillation product σ as seen above. Nevertheless there is a minor difference: since ρ has rank two, we can now find a *direct* transformation $\rho \mapsto \sigma$, which clearly must be given by the operation elements:

$$A \otimes B := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}.$$

□

We have seen that the appropriate method to study single-copy distillation procedures is looking at the entanglement properties of the state's support. These properties decide whether a state can be brought directly into Bell-diagonal form having optimal entanglement of formation or whether this can be achieved only in a limiting process with vanishing probability in the limit. It is an obvious task to generalize these ideas to higher dimension. It must be mentioned, nevertheless, that concepts like *unextendible product bases* will play a major role and will make such an analysis far more sophisticated.

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